

Lecture 2

Reminder: No class this Thursday (^{± will}_{do a} make-up)

OH change: Now wed/fri + by appointment

Two illustrations of OT.

EX 2. Parabolic PDE and
minimax moment

The porous medium eqn is an important

PDE: for $m \geq 1$ it is given by

$$\partial_t \rho = \Delta \rho^m$$

For $m=1$ this gives back the H.E.

Fix $\rho_0 \in L^1(\mathbb{R}^d)$, $\rho_0 \geq 0$, fix $h > 0$

$$\min_P \left\{ h \int \rho^m dx + \frac{1}{2} d_2(P, \rho_0) \right\}$$

... $p_0, p_1, p_2, \dots$ Define $p^{(n)}(x, t)$

Theorem (Otto ~ 2000)

The sequence $p^{(n)} \rightarrow p$, very weak
sol. to $\partial_t p = \Delta p^m$, $p(x, 0) = p_0(x)$

A follow up paper by JKO

connect the sum for Fokker-Planck:

$$\partial_t p = \operatorname{div}(\nabla p + p \nabla V) \quad (\text{FP})$$

If $p_\infty := \frac{1}{Z} e^{-V}$, then

$$\min_p \left\{ h \int p \log(p/p_\infty) dx + \frac{1}{2} d_2^2(p, p_\infty) \right\}$$

Produce a rate for FP...

If V is α -conv, then

$$KL(p(t), p_\infty) \leq e^{-\alpha t} h_2(p_0, p_\infty)$$

...

Relevant to: Kinetic theory
Monte-Carlo ...

EX 3: OT and curv on
a Riemann manifold

... Ricci curv!

Displacement integrals: (McCann)

Given two probs μ_0, μ_1

$\exists!$ $t \rightarrow \mu_t \in \mathcal{P}_2(M)$ s.t.

$$d_2(\mu_t, \mu_0) = t \quad d(\mu_0, \mu_1)$$

(a geodesic)

Lott - Villani / Sturm:

A mfd has $\text{Ric} \geq 0 \iff$

$$H(p) = \int_M p \log p \, d\mu$$

is displacement - can

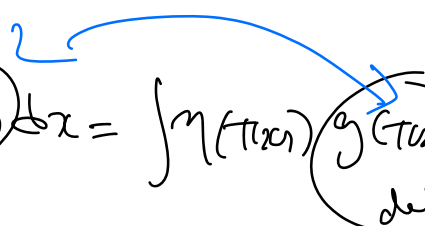
A word from our sponsor:
the Monge-Ampère equation

Suppose $T = Du$ is smooth ... and

$$\int_{\mathbb{R}^n} \eta(T(x)) f(x) dx = \int \eta(y) g(y) dy$$

✓ q. -

Then

$$\int \eta(T(x)) \underbrace{f(x)}_{\text{circled}} dx = \int \eta(T(x)) \underbrace{g(T(x))}_{\text{circled}} \det DT dx$$


$$\Rightarrow \det(DT(x)) = \frac{f(x)}{g(T(x))}$$

$$\text{or, } \boxed{\det(D^2 u(x)) = \frac{f(x)}{g(Du(x))}}$$

Compare with Poisson:

$$\Delta u(x) = f(x) \dots$$

Regularity of optimal transport maps ~~is~~ regular of solution to the Monge-Ampère eqn.

The Monge-Kantorovich problem

Now you describe the

Monge-Kantorovich problem

for two spaces X, Y

(say, compact metric spaces)

Given μ, ν , let $C: X \times Y \rightarrow \mathbb{R}$
cost.

Consider two ^{Bad} problems
 μ, ν in X, Y .

A transport plan π on $X \times Y$ is a prob
str.

$$\int_{X \times Y} \phi(x, y) d\pi(x, y) = \int_X \phi(x) d\mu(x)$$

$$\int_{X \times Y} \psi(y) d\pi(x, y) = \int_Y \psi(y) d\nu(y)$$

$$\triangleright \phi \in C_c(X), \psi \in C_c(Y)$$

Example: Given $T: X \rightarrow Y$
and $\mu \in \mathcal{P}(X)$, define π by

$$\int \phi(x, y) d\pi(x, y) := \int \phi(x, T(x)) d\mu(x)$$

Observe,

$$\int \phi(x) d\pi(x(y)) = \int \phi(x) d\mu_x,$$

$$\int \psi(y) d\pi(x(y)) = \int \psi(T(x)) d\mu_x,$$

So a Boole transform map gives rise to a transfer map.